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Extremes of space–time Gaussian processes

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Abstract

Let $Z = \{Z_t(h); h \in \mathbb{R}^d, t \in \mathbb{R}\}$ be a space–time Gaussian process which is stationary in the time variable t. We study $M_n(h) = \sup_{t\in[0,n]} Z_t(s_n h)$, the supremum of Z taken over $t \in [0, n]$ and rescaled by a properly chosen sequence $s_n \to 0$. Under appropriate conditions on Z, we show that for some normalizing sequence $b_n \to \infty$, the process $b_n(M_n - b_n)$ converges as $n \to \infty$ to a stationary max-stable process of Brown–Resnick type. Using strong approximation, we derive an analogous result for the empirical process.

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1. Introduction and statement of results

1.1. Introduction

Let X_i, $i \in \mathbb{N}$, be independent copies of a Gaussian process $\{X(h); h \in D\}$. We suppose that X has zero mean, unit variance, continuous sample paths, and is defined on $D \subset \mathbb{R}^d$, an open set containing the origin. Further, we suppose that the covariance function of X, $r^X(h_1, h_2) = \mathbb{E}[X(h_1)X(h_2)]$, satisfies the following condition: for some $\alpha \in (0, 2]$ and $c_\alpha > 0$,

(X1) $r^X(\varepsilon h_1, \varepsilon h_2) = 1 - c_\alpha|h_1 - h_2|^\alpha \varepsilon^\alpha + o(\varepsilon^\alpha)$ as $\varepsilon \downarrow 0$, where the o-term is uniform in $h_1, h_2 \in D$.

Here, $|\cdot|$ denotes the Euclidian norm on $\mathbb{R}^d$.

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The limiting properties, as $n \to \infty$, of the maximum of $X_1, \dots, X_n$, taken pointwise, were studied by Brown and Resnick [3] and Kabluchko et al. [13]. It was observed in [3] that in order to obtain a nontrivial limiting process, an additional spatial rescaling need to be introduced. We need normalizing sequences s_n and b_n defined by

$$s_n = \frac{1}{(2c_\alpha \log n)^{1/\alpha}}, \tag{1}$$

$$b_n = \sqrt{2\log n} - \frac{1}{\sqrt{2\log n}}\left(\frac{1}{2}\log\log n + \log(2\sqrt{\pi})\right). \tag{2}$$

Define a stochastic process $\{M_n(h); h \in s_n^{-1}D\}$, where $s_n^{-1}D$ denotes the set $\{s_n^{-1}h : h \in D\}$, by

$$M_n(h) = \max_{i=1,\dots,n} X_i(s_n h). \tag{3}$$

Then it follows from a more general result of [13, Theorem 17] that the process M_n^* defined by $M_n^*(h) = b_n(M_n(h) - b_n)$ converges as $n \to \infty$ to some nontrivial limiting process η_α weakly on $C(K)$, the space of continuous functions on any fixed compact set $K \subset \mathbb{R}^d$. If X is the Ornstein–Uhlenbeck process on $\mathbb{R}$ with covariance function $r^X(h_1, h_2) = \mathrm{e}^{-|h_1-h_2|}$, the above result is due to Brown and Resnick [3], the limit being η_1. Other particular cases, leading to the process η_2, were considered in [8,9]. Closely related results were obtained by Pickands [18,19] and Hüsler and Reiss [11]. Applications were given in [7,4].

The limiting process η_α will be called the Brown–Resnick process with parameter $\alpha \in (0, 2]$, and can be described as follows. Let $\{U_i\}_{i=1}^\infty$ be an enumeration of the points of a Poisson point process with intensity $\mathrm{e}^{-u}\mathrm{d}u$ on $\mathbb{R}$. Further, let W_i, $i \in \mathbb{N}$, be independent copies of a drifted (Lévy) fractional Brownian motion $\{W(h); h \in \mathbb{R}^d\}$ with

$$\mathrm{Cov}(W(h_1), W(h_2)) = |h_1|^\alpha + |h_2|^\alpha - |h_1 - h_2|^\alpha, \tag{4}$$

$$\mathbb{E}[W(h)] = -|h|^\alpha. \tag{5}$$

Then the Brown–Resnick process $\{\eta_\alpha(h); h \in \mathbb{R}^d\}$ is defined by

$$\eta_\alpha(h) = \max_{i\in\mathbb{N}}(U_i + W_i(h)). \tag{6}$$

The process η_α is stationary (although this is not evident from Eq. (6)), sample continuous, with unit Gumbel margins, see [13] for more properties.

1.2. Main result

In this paper, we study the limiting behavior of the supremum, taken over continuous time and considered as a function of space, of a space–time Gaussian process. Let us be more precise. Let $Z = \{Z_t(h); h \in D, t \in \mathbb{R}\}$ be a zero mean and unit variance sample continuous Gaussian process which is stationary in the time variable t. Here, D is an open subset of $\mathbb{R}^d$ containing 0. The covariance function of Z, $r_t(h_1, h_2) = \mathbb{E}[Z_s(h_1)Z_{s+t}(h_2)]$, does not depend on s by the time stationarity.

We suppose that the following conditions are satisfied for some $\alpha, \beta \in (0, 2]$ and $c_\alpha, c_\beta > 0$:

(Z1) $r_{\varepsilon^{1/\beta}t}(\varepsilon^{1/\alpha}h_1, \varepsilon^{1/\alpha}h_2) = 1 - (c_\alpha|h_1 - h_2|^\alpha + c_\beta|t|^\beta)\varepsilon + o(\varepsilon)$ as $\varepsilon \downarrow 0$, where the o-term is uniform as long as $h_1, h_2 \in D$ and t stays bounded.

(Z2) $r_t(h_1, h_2) < 1$ provided that $t \neq 0$, $h_1, h_2 \in D$.

(Z3) $r_t(h_1, h_2) = o(1/\log|t|)$ as $t \to \infty$ uniformly in $h_1, h_2 \in D$.

Define

$$s_n = \frac{1}{(2c_\alpha \log n)^{1/\alpha}}, \tag{7}$$

$$b_n = \sqrt{2\log n} + \frac{1}{\sqrt{2\log n}} \left(\frac{2-\beta}{2\beta} \log\log n + \log\left(\frac{(2c_\beta)^{\frac{1}{\beta}} H_\beta}{2\sqrt{\pi}} \right) \right). \tag{8}$$

Here, $H_\beta > 0$ is the so-called Pickands constant [18,19], see Eqs. (28), (30) below.

In the next theorem, which is our main result, we are interested in the limiting behavior, as $n \to \infty$, of the process $\{M_n(h); h \in s_n^{-1} D\}$ defined by

$$M_n(h) = \sup_{t \in [0,n]} Z_t(s_n h). \tag{9}$$

Theorem 1.1. *The process M_n^* defined by $M_n^*(h) = b_n(M_n(h) - b_n)$ converges as $n \to \infty$ to the Brown–Resnick process η_α weakly on $C(K)$ for every compact set $K \subset \mathbb{R}^d$.*

The process M_n^* is defined on the domain $s_n^{-1} D$ which contains any fixed compact set K if n is large enough. Therefore, the restriction of M_n^* to K is indeed well defined provided that n is large enough.

1.3. Remarks

Remark 1.1. For a fixed $h \in \mathbb{R}^d$, the limiting distribution of $M_n^*(h)$ was determined by Pickands [18,19] who showed that

$$\lim_{n\to\infty} \mathbb{P}\left[M_n^*(h) \le \tau\right] = \exp(-\mathrm{e}^{-\tau}) \quad \text{for all } \tau \in \mathbb{R}. \tag{10}$$

Our Theorem 1.1 can be viewed as a functional version of Pickands' result.

Remark 1.2. The extreme values of a class of space–time processes with *heavy tails* were studied by Davis and Mikosch [6]. The results and methods of [6] are completely different from ours.

Remark 1.3. Introducing a normalizing sequence s_n into (9), which results in spatial rescaling of the process under consideration, is necessary to obtain a limit with nontrivial dependence between margins and was suggested in [3,11]. In fact, the results of [23,16,17,1,10,12] show that the maxima of two or more dependent stationary Gaussian processes, taken in continuous or discrete time over the interval $[0, n]$, become asymptotically independent as $n \to \infty$ under rather general conditions on the dependence between the processes (these conditions, however, do not allow the dependence to get stronger as $n \to \infty$). It can be shown that up to a multiplicative constant, the sequence s_n as defined in (7) is the only sequence which leads to a nontrivial limiting process for M_n.

Remark 1.4. The appearance of the Brown–Resnick process η_α as the limit of the *discrete-time* maximum M_n defined in (3) has a natural explanation: it is known that

(a) the properly normalized point process formed by the extremes of the sequence $X_1(0), \ldots, X_n(0)$ converges as $n \to \infty$ to the Poisson point process $\{U_i\}_{i=1}^\infty$ with intensity $\mathrm{e}^{-u}\mathrm{d}u$ (see [22, Section 4.4.2]);

(b) the behavior of the process X_i conditioned on the event "$X_i(0)$ is large" is described, in an appropriate sense, by the drifted fractional Brownian motion W defined in (4) and (5) (see [18]).

A remarkable feature of Theorem 1.1 is that the same process η_α, constructed starting with a *countable* number of "extremes" U_i, arises as the limit of the *continuous-time* supremum (9). This may be explained as follows. Suppose that the process $\{Z_t(0); t \in \mathbb{R}\}$ takes only extremely large values in some small interval I and let $t_0 = \arg\sup_{t\in I} Z_t(0)$. Then the "local direct sum" structure of Condition (Z1) implies that $\sup_{t\in I} Z_t(s_n h) \approx Z_{t_0}(s_n h)$ for large n as long as h stays bounded. Thus, t_0 is essentially the only point in the interval I which has a chance to contribute to the global supremum (9). In the limit $n \to \infty$, countably many such points emerge.

1.4. Processes with product-form covariance

Theorem 1.1 applies to a class of Gaussian processes with product-form covariance. More precisely, suppose that $\{X(h); h \in D\}$ satisfies Condition (X1), and let $\{Y(t); t \in \mathbb{R}\}$ be a stationary sample continuous zero-mean Gaussian process with covariance function $r^Y(t) = \mathbb{E}[Y(0)Y(t)]$ satisfying the following three conditions for some $\beta \in (0, 2]$ and $c_\beta > 0$:

(Y1) $r^Y(t) = 1 - c_\beta|t|^\beta + o(|t|^\beta)$ as $t \to 0$.
(Y2) $r^Y(t) < 1$ provided that $t \neq 0$.
(Y3) $r^Y(t) = o(1/\log|t|)$ as $t \to \infty$.

Given X and Y, we construct a zero-mean time-stationary Gaussian process $Z = \{Z_t(h); h \in D, t \in \mathbb{R}\}$ with covariance

$$r_t(h_1, h_2) = r^X(h_1, h_2) r^Y(t). \tag{11}$$

The process Z, which may be thought of as a dynamical version of the spatial process X, is easily seen to satisfy Conditions (Z1)–(Z3). An example is given by the two-dimensional generalized Ornstein–Uhlenbeck process with covariance function

$$r_t(h_1, h_2) = \exp\left\{-c_\alpha|h_1 - h_2|^\alpha - c_\beta|t|^\beta\right\}.$$

1.5. Application to the empirical process

Let us mention an application of Theorem 1.1 to the empirical process. Let $V_1, V_2, \dots$ be i.i.d. random variables distributed uniformly on the interval [0, 1]. The empirical process $\{\alpha_t(h); h \in [0, 1], t \geq 1\}$ is defined by

$$\alpha_t(h) = \sqrt{t}\left(\frac{1}{[t]}\sum_{i=1}^{[t]} \mathbf{1}_{\{V_i \leq h\}} - h\right).$$

Fix $h_0 \in (0, 1)$. With the notation $\log_2 n = \log\log n$, $\log_3 n = \log\log\log n$, we define

$$\tilde{s}_n = \frac{h_0(1-h_0)}{\log_2 n}, \tag{12}$$

$$\tilde{b}_n = \sqrt{2\log_2 n} + \frac{1}{\sqrt{2\log_2 n}}\left(\frac{1}{2}\log_3 n - \log(2\sqrt{\pi})\right). \tag{13}$$

We are interested in the maximum of the empirical process $\alpha_t(h)$ taken over the set of "times" $t \in [1, n]$ and considered as a function of the "space" variable h. More precisely, we define

$$L_n(h) = \frac{1}{\sqrt{h_0(1-h_0)}} \sup_{t\in[1,n]} \alpha_t(h_0 + \tilde{s}_n h). \tag{14}$$

Theorem 1.2. *The process $\tilde{b}_n(L_n - \tilde{b}_n)$ converges as $n \to \infty$ to the drifted Brown–Resnick process $\eta = \{\eta_1(h) + (1-2h_0)h;\ h \in \mathbb{R}\}$ in the sense of finite-dimensional distributions.*

Note that the drift $(1-2h_0)$ in the above theorem is positive for $h_0 < 1/2$ and negative for $h_0 > 1/2$, which agrees with the intuition that the empirical process tends to increase on $[0, 1/2]$ and to decrease on $[1/2, 1]$. Further, the convergence of the finite-dimensional distributions can be strengthened to the weak convergence on $C(K)$ for every compact set $K \subset \mathbb{R}$ if one modifies the definition of the empirical process so that it becomes continuous.

1.6. Extensions of the main result

Theorem 1.1 can be extended in several directions. First, note that Condition (Z1) implies a sort of local isotropy for the space part of the process Z. An appropriate version of Theorem 1.1 holds if Condition (Z1) is replaced by a more general regular variation assumption which is similar to Assumption 16 in [13] and does not require local isotropy.

Second, it is possible to show that the process $M_n^*(h)$, considered as a *space–time* process, converges to a space–time version of the Brown–Resnick process η_α. Before we can state this result, we need to describe the limiting process. Informally, it is a direct product of the usual Brown–Resnick process η_α and the extremal process of Dwass–Lamperti (see [22, Chapter 4] for the definition of the extremal process). More precisely, let $\{(U_i, T_i)\}_{i=1}^\infty$ be an enumeration of the points of a Poisson point process on $\mathbb{R} \times [0, \infty)$ with intensity $\mathrm{e}^{-u}\mathrm{d}u \times dt$. Let W_i, $i \in \mathbb{N}$, be independent copies of a drifted (Lévy) fractional Brownian motion $\{W(h);\ h \in \mathbb{R}^d\}$ defined as in (4) and (5). Then the space–time Brown–Resnick process $\{\eta_\alpha(h;t);\ h \in \mathbb{R}^d, t \geq 0\}$ is defined by

$$\eta_\alpha(h;t) = \max_{i\in\mathbb{N}: T_i \in [0,t]} (U_i + W_i(h)).$$

Note that the restriction of this process to $t = 1$ has the same law as the usual Brown–Resnick process, and that for fixed $h \in \mathbb{R}^d$, the process $\{\eta_\alpha(h;t);\ t \geq 0\}$ has the law of the extremal process of Dwass–Lamperti. The space–time version of Theorem 1.1 reads as follows.

Theorem 1.3. *Under the assumptions of Section 1.2, the process $\{M_n^*(h;t);\ h \in s_n^{-1}D, t \geq 0\}$ defined by $M_n^*(h;t) = b_n(M_{tn}(h) - b_n)$ converges as $n \to \infty$ to the space–time Brown–Resnick process $\{\eta_\alpha(h;t);\ h \in \mathbb{R}^d, t \geq 0\}$ in the sense of finite-dimensional distributions.*

We omit the proof of Theorem 1.3 since it is based on the same ideas as the proof of Theorem 1.1.

1.7. Organization of the paper

In Section 2 we prove Theorem 1.1. In Section 3 we deduce Theorem 1.2 from Theorem 1.1 using a strong invariance principle connecting the empirical process to the Kiefer process.

2. Proof of Theorem 1.1

2.1. Convergence of finite-dimensional distributions

Our main goal in this section is to prove Proposition 2.1 below which shows the convergence of finite-dimensional distributions in Theorem 1.1. We always use the notation of Section 1.2.

Proposition 2.1. *Let $k \in \mathbb{N}$, and fix $h_1, \dots, h_k \in \mathbb{R}^d$ and $\tau_1, \dots, \tau_k \in \mathbb{R}$. Define*

$$P_n = \mathbb{P}\left[M_n^*(h_i) \le \tau_i \text{ for all } i = 1, \dots, k\right]. \tag{15}$$

Let $\{\eta_\alpha(h); h \in \mathbb{R}^d\}$ be the Brown–Resnick process with parameter α. Then

$$\lim_{n\to\infty} P_n = \mathbb{P}\left[\eta_\alpha(h_i) \le \tau_i \text{ for all } i = 1, \dots, k\right]. \tag{16}$$

Let $\{B_\alpha(x); x \in \mathbb{R}^d\}$ and $\{B_\beta(x); x \in \mathbb{R}\}$ be drifted (Lévy) fractional Brownian motions with

$$\operatorname{Cov}(B_\alpha(x_1), B_\alpha(x_2)) = |x_1|^\alpha + |x_2|^\alpha - |x_1 - x_2|^\alpha, \qquad \mathbb{E}[B_\alpha(x)] = -|x|^\alpha, \tag{17}$$

$$\operatorname{Cov}(B_\beta(x_1), B_\beta(x_2)) = |x_1|^\beta + |x_2|^\beta - |x_1 - x_2|^\beta, \qquad \mathbb{E}[B_\beta(x)] = -|x|^\beta. \tag{18}$$

We always assume that B_α and B_β are independent.

Let $G_{h,\tau} = G_{\{h_i\}_{i=1}^k, \{\tau_i\}_{i=1}^k}$ be a constant defined by

$$G_{h,\tau} = \mathbb{E} \exp\left\{\max_{i=1,\dots,k} (B_\alpha(h_i) - \tau_i)\right\}.$$

By [13, Eq. (5)], the finite-dimensional distributions of the process η_α are given by

$$\mathbb{P}\left[\eta_\alpha(h_i) \le \tau_i \text{ for all } i = 1, \dots, k\right] = \mathrm{e}^{-G_{h,\tau}}. \tag{19}$$

Hence, to prove Proposition 2.1, we have to show that

$$\lim_{n\to\infty} P_n = \mathrm{e}^{-G_{h,\tau}}. \tag{20}$$

The rest of Section 2.1 is devoted to the proof of (20). We will use a multivariate version of the method of Pickands [18,19]. We always refer to Leadbetter et al. [14, Chapter 12] and Piterbarg [20, Section D] if we need facts proved by Pickands. Sometimes, we omit technical details in order to avoid repetition.

First we fix the notation. We take some $a > 0$ and define $q_n = a c_\beta^{-1/\beta} b_n^{-2/\beta}$. We will start by analyzing the extreme-value behavior of the process Z over a small time interval $[0, q_n]$. To this end, we define a process $\{Z_t^*(h); h \in s_n^{-1} D, t \in \mathbb{R}\}$ dependent also on n by

$$Z_t^*(h) = b_n(Z_{q_n t}(s_n h) - b_n).$$

For $w \in \mathbb{R}$ and $n \in \mathbb{N}$, let $Z^{w,n} = \{Z_t^{w,n}(h); h \in s_n^{-1} D, t \in \mathbb{R}\}$ be the process $\{Z_t^*(h) - Z_0^*(0); h \in s_n^{-1} D, t \in \mathbb{R}\}$ conditioned on the event $Z_0^*(0) = w$.

Lemma 2.1. *Fix a cube $K = [-A, A]^d$ and let B_α and B_β be independent fractional Brownian motions as in (17), (18).*

1. *Let $w \in \mathbb{R}$ be fixed. Then, as $n \to \infty$, we have*

$$\{Z_t^{w,n}(h); h \in K, t \in [0, 1]\} \Rightarrow \{B_\alpha(h) + B_\beta(at); h \in K, t \in [0, 1]\} \tag{21}$$

   weakly on $C(K \times [0, 1])$.

2. *If N is sufficiently large, then the family of processes*

$$\{Z^{w,n} - \mathbb{E} Z^{w,n}\}_{w \in \mathbb{R}, n > N}$$

   is tight on $C(K \times [0, 1])$.

3. *If N is sufficiently large, then the family of processes*

$$\{Z^{w,n}\}_{w\in[-c,c],n>N}$$

is tight on $C(K\times[0,1])$ for every $c>0$.

Proof. Take $h, h_1, h_2 \in K$ and $t, t_1, t_2 \in [0,1]$. The well-known formulas for the conditional Gaussian distributions show that the process $Z^{w,n}$ is Gaussian with expectation

$$\mathbb{E}[Z_t^{w,n}(h)] = -(b_n^2 + w)(1 - r_{q_n t}(s_n h, 0)) \tag{22}$$

and covariance function

$$\mathrm{Cov}(Z_{t_1}^{w,n}(h_1), Z_{t_2}^{w,n}(h_2)) = b_n^2(r_{q_n(t_1-t_2)}(s_n h_1, s_n h_2) - r_{q_n t_1}(s_n h_1, 0) r_{q_n t_2}(0, s_n h_2)). \tag{23}$$

Note that $q_n t = b_n^{-2/\beta} \cdot (a c_\beta^{-1/\beta} t)$ and $s_n h_i \sim b_n^{-2/\alpha} \cdot (c_\alpha^{-1/\alpha} h_i)$, $i = 1, 2$, as $n \to \infty$. Condition (Z1) with $\varepsilon = b_n^{-2}$ implies that, as $n \to \infty$,

$$r_{q_n t}(s_n h_1, s_n h_2) = 1 - b_n^{-2}(|h_1 - h_2|^\alpha + a^\beta |t|^\beta)(1 + o(1)), \tag{24}$$

where the o-term is uniform in $h_1, h_2 \in K$, $t \in [0,1]$.

Applying (24) to (22) yields

$$\lim_{n\to\infty} \mathbb{E}[Z_t^{w,n}(h)] = -(|h|^\alpha + a^\beta |t|^\beta). \tag{25}$$

Similarly, applying (24) to (23), we obtain that uniformly in $h_1, h_2 \in K$ and $t_1, t_2 \in [0,1]$,

$$\begin{aligned}&\lim_{n\to\infty} \mathrm{Cov}(Z_{t_1}^{w,n}(h_1), Z_{t_2}^{w,n}(h_2))\\ &\quad= (|h_1|^\alpha + |h_2|^\alpha - |h_1 - h_2|^\alpha) + a^\beta(|t_1|^\beta + |t_2|^\beta - |t_1 - t_2|^\beta).\end{aligned} \tag{26}$$

This shows the convergence of finite-dimensional distributions in (21).

Since the weak convergence in (21) will follow from Part 3 of the lemma, we proceed to the proof of Part 2. We have, applying (23),

$$\begin{aligned}&\mathrm{Var}(Z_{t_1}^{w,n}(h_1) - Z_{t_2}^{w,n}(h_2))\\ &\quad= b_n^2\left(2 - 2r_{q_n(t_1-t_2)}(s_n h_1, s_n h_2) - (r_{q_n t_1}(s_n h_1, 0) - r_{q_n t_2}(0, s_n h_2))^2\right)\\ &\quad\le b_n^2\left(2 - 2r_{q_n(t_1-t_2)}(s_n h_1, s_n h_2)\right).\end{aligned}$$

By (24), this implies that as $n \to \infty$,

$$\mathrm{Var}(Z_{t_1}^{w,n}(h_1) - Z_{t_2}^{w,n}(h_2)) \le (2 + o(1))(|h_1 - h_2|^\alpha + a^\beta |t_1 - t_2|^\beta). \tag{27}$$

Note that the left-hand side of (27) does not depend on w, see (23). It follows that for $n > N$ and all $w \in \mathbb{R}$,

$$\mathrm{Var}(Z_{t_1}^{w,n}(h_1) - Z_{t_2}^{w,n}(h_2)) < 3(|h_1 - h_2|^\alpha + a^\beta |t_1 - t_2|^\beta).$$

Together with $Z_0^{w,n}(0) = 0$, this implies that the family of processes $\{Z^{w,n} - \mathbb{E}Z^{w,n}\}_{w\in\mathbb{R},n>N}$ is tight (this fact goes back to Kolmogorov, see e.g. [15, Corollary 11.7]). This proves Part 2 of the lemma.

To prove Part 3 of the lemma, note that (22) implies that the convergence in (25) is uniform in $w \in [-c, c]$, where $c > 0$ is fixed. It follows that the family of functions $\{\mu^{w,n}\}_{w\in[-c,c],n>N}$, where $\mu^{w,n}(h,t) = \mathbb{E}[Z_t^{w,n}(h)]$, is weakly precompact in $C(K \times [0,1])$. Recalling Part 2 of the lemma, we obtain that the family of processes $\{Z^{w,n}\}_{w\in[-c,c],n>N}$ is tight. $\square$

It will be convenient to write $u_{in} = b_n + b_n^{-1}\tau_i$. Let $m \in \mathbb{N}$ and $a > 0$ be fixed. Define a constant

$$H_\beta(m, a) = \mathbb{E} \exp\left\{ \max_{j=0,1,\ldots,m-1} B_\beta(aj) \right\}. \tag{28}$$

Lemma 2.2. *Let*

$$p_n(m, a) = \mathbb{P}\left[\max_{j=0,\ldots,m-1} Z_{jq_n}(s_n h_i) > u_{in} \textit{ for some } i = 1, \ldots, k \right].$$

Then the following asymptotic relation holds true as $n \to \infty$:

$$p_n(m, a) \sim G_{h,\tau} H_\beta(m, a)(\sqrt{2\pi})^{-1} b_n^{-1} \mathrm{e}^{-b_n^2/2}.$$

Proof. In the subsequent equations the indices i and j range over $1, \ldots, k$ and $0, \ldots, m-1$, respectively. The density of the random variable $Z_0^*(0)$ is given by

$$f_{Z_0^*(0)}(w) = (2\pi)^{-1/2} b_n^{-1} \mathrm{e}^{-(b_n + b_n^{-1} w)^2/2} \mathrm{d}w. \tag{29}$$

Conditioning on $Z_0^*(0) = w$, we obtain

$$\begin{aligned} p_n(m, a) &= \mathbb{P}\left[\exists i, j : Z_j^*(h_i) > \tau_i\right] \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbb{P}\left[\exists i, j : Z_j^*(h_i) > \tau_i | Z_0^*(0) = w\right] \mathrm{e}^{-(b_n + b_n^{-1} w)^2/2} b_n^{-1} \mathrm{d}w \\ &= \frac{1}{\sqrt{2\pi}} b_n^{-1} \mathrm{e}^{-b_n^2/2} \int_{-\infty}^{\infty} \mathbb{P}[\exists i, j : Z_j^{w,n}(h_i) > \tau_i - w] \mathrm{e}^{-w} \mathrm{e}^{-\frac{w^2}{2b_n^2}} \mathrm{d}w. \end{aligned}$$

Applying Lemma 2.1 to the probability under the integral sign and omitting the standard justification of the use of the dominated convergence theorem, we obtain that

$$\begin{aligned} &\lim_{n\to\infty} \sqrt{2\pi} b_n \mathrm{e}^{b_n^2/2} p_n(m, a) \\ &= \int_{-\infty}^{\infty} \mathbb{P}\left[\max_{i=1,\ldots,k} (B_\alpha(h_i) - \tau_i) + \max_{j=0,\ldots,m-1} B_\beta(aj) > -w \right] \mathrm{e}^{-w} \mathrm{d}w \\ &= \mathbb{E} \exp\left\{ \max_{i=1,\ldots,k} (B_\alpha(h_i) - \tau_i) + \max_{j=0,\ldots,m-1} B_\beta(aj) \right\} \\ &= G_{h,\tau} H_\beta(m, a). \end{aligned}$$

This completes the proof of Lemma 2.2. $\square$

Fix $l > 0$. In our next step, we are interested in the asymptotic behavior of the probability

$$Q_n(a) = \mathbb{P}\left[\max_{t \in [0,l] \cap q_n \mathbb{Z}} Z_t(s_n h_i) > u_{in} \text{ for some } i = 1, \ldots, k \right].$$

It was shown in [14, Lemmas 12.2.4, 12.2.7, 12.2.8] that the limits in the following formula exist finitely and are strictly positive:

$$H_\beta(a) = \lim_{m\to\infty} \frac{1}{ma} H_\beta(m, a), \qquad H_\beta = \lim_{a\to 0} H_\beta(a). \tag{30}$$

Lemma 2.3. *The following asymptotic equality holds as $n \to \infty$:*

$$Q_n(a) \sim l H_\beta(a) H_\beta^{-1} G_{h,\tau} n^{-1}. \tag{31}$$

Proof. Since the proof uses the "double sum" method of Pickands, see [14, Lemma 12.2.4], we omit some details. Define a random event $A_n(m,a,t)$ by

$$A_n(m,a,t) = \left\{ \max_{j=0,\dots,m-1} Z_{t+jq_n}(s_n h_i) > u_{in} \text{ for some } i = 1,\dots,k \right\}.$$

By the Bonferroni inequality, we have

$$Q_n(a) \le S_n'(m,a), \qquad Q_n(a) \ge S_n'(m,a) - S_n''(m,a), \tag{32}$$

where

$$S_n'(m,a) = \sum_{t \in [0,l] \cap mq_n\mathbb{Z}} \mathbb{P}[A_n(m,a,t)], \tag{33}$$

$$S_n''(m,a) = \sum_{\substack{t_1,t_2 \in [0,l] \cap mq_n\mathbb{Z} \\ t_1 \ne t_2}} \mathbb{P}[A_n(m,a,t_1) \cap A_n(m,a,t_2)]. \tag{34}$$

There are $[l/(mq_n)] + 1$ terms to on the right-hand side of (33), and these terms are equal since the process Z is stationary in time. Applying Lemma 2.2 to each of these terms, we obtain

$$S_n'(m,a) \sim \frac{l}{mq_n} \cdot G_{h,\tau} H_\beta(m,a) (\sqrt{2\pi})^{-1} b_n^{-1} \mathrm{e}^{-b_n^2/2}, \quad n \to \infty. \tag{35}$$

Recalling that $q_n = a c_\beta^{-1/\beta} b_n^{-2/\beta}$, we may rewrite this as

$$S_n'(m,a) \sim l G_{h,\tau} \cdot \left(\frac{1}{ma} H_\beta(m,a) \right) \cdot \left(c_\beta^{1/\beta} (\sqrt{2\pi})^{-1} b_n^{(2/\beta)-1} \mathrm{e}^{-b_n^2/2} \right), \quad n \to \infty. \tag{36}$$

An easy calculation based on (8) shows that

$$c_\beta^{1/\beta} (\sqrt{2\pi})^{-1} b_n^{(2/\beta)-1} \mathrm{e}^{-b_n^2/2} \sim H_\beta^{-1} n^{-1}, \quad n \to \infty. \tag{37}$$

Applying this to (36), we obtain

$$S_n'(m,a) \sim l G_{h,\tau} \cdot \left(\frac{1}{ma} H_\beta(m,a) \right) \cdot H_\beta^{-1} n^{-1}, \quad n \to \infty. \tag{38}$$

The double sum $S_n''(m,a)$ can be bounded as in the proof of Lemma 12.2.4 of [14]:

$$\limsup_{m\to\infty} \limsup_{n\to\infty} n S_n''(m,a) = 0. \tag{39}$$

The statement of the lemma follows by letting $m \to \infty$ in (32) combined with (38) and (39). $\square$

Fix $\varepsilon > 0$. For $j = 0, 1, \dots$, denote by I_j the interval $[j, j+1-\varepsilon]$. Further, define

$$P_n(a,\varepsilon) = \mathbb{P}\left[\max_{t \in (\cup_{j=0}^{n-1} I_j) \cap q_n\mathbb{Z}} Z_t(s_n h_i) \le u_{in} \text{ for all } i = 1,\dots,k \right].$$

Lemma 2.4. *Let $\rho_{a,\varepsilon} = \limsup_{n\to\infty}(P_n(a,\varepsilon) - P_n)$. Then $\lim_{a,\varepsilon\to 0} \rho_{a,\varepsilon} = 0$.*

Proof. It is clear that $\rho_{a,\varepsilon} \geq 0$. For $i = 1, \dots, k$, let $\rho_{a,\varepsilon}^{(i)}$ be defined as

$$\limsup_{n\to\infty} \left(\mathbb{P}\left[\max_{t\in(\cup_{j=0}^{n-1} I_j)\cap q_n\mathbb{Z}} Z_t(s_n h_i) \leq u_{in} \right] - \mathbb{P}\left[\sup_{t\in[0,n]} Z_t(s_n h_i) \leq u_{in} \right] \right).$$

By [14, Lemma 12.3.2], we have $\lim_{a,\varepsilon\to 0} \rho_{a,\varepsilon}^{(i)} = 0$. The statement of the lemma follows by noting that $\rho_{a,\varepsilon} \leq \sum_{i=1}^{k} \rho_{a,\varepsilon}^{(i)}$. $\square$

Lemma 2.5. *We have*

$$\lim_{n\to\infty} P_n(a, \varepsilon) = \exp(-(1-\varepsilon)H_\beta(a)H_\beta^{-1}G_{h,\tau}).$$

Proof. Given $t_1, t_2 \geq 0$, we write $t_1 \sim t_2$ if there is $j = 0, 1, \dots$ with $t_1, t_2 \in I_j$. Otherwise, we write $t_1 \nsim t_2$. Let $\{U_{t,i}^{(n)}; t \in (\cup_{j=0}^{n-1} I_j) \cap q_n\mathbb{Z}, i = 1, \dots, k\}$ be a zero-mean Gaussian vector with the following covariance structure:

$$\mathbb{E}[U_{t_1,i_1}^{(n)} U_{t_2,i_2}^{(n)}] = \begin{cases} r_{t_1-t_2}(s_n h_{i_1}, s_n h_{i_2}), & \text{if } t_1 \sim t_2, \\ 0, & \text{if } t_1 \nsim t_2. \end{cases} \tag{40}$$

It follows from (40) that

$$\mathbb{P}\left[\forall i : \max_{t\in(\cup_{j=0}^{n-1} I_j)\cap q_n\mathbb{Z}} U_{t,i}^{(n)} \leq u_{in} \right] = \left(1 - \mathbb{P}\left[\exists i : \max_{t\in I_0\cap q_n\mathbb{Z}} U_{t,i}^{(n)} > u_{in} \right] \right)^n.$$

(Here, i ranges over $1, \dots, k$). Applying Lemma 2.3 to the probability on the right-hand side, we get

$$\lim_{n\to\infty} \mathbb{P}\left[\forall i : \max_{t\in(\cup_{j=0}^{n-1} I_j)\cap q_n\mathbb{Z}} U_{t,i}^{(n)} \leq u_{in} \right] = \exp(-(1-\varepsilon)H_\beta(a)H_\beta^{-1}G_{h,\tau}). \tag{41}$$

To complete the proof, we need to show that $P_n(a, \varepsilon)$ is close to the probability on the left-hand side of (41). More precisely, we will show that

$$\lim_{n\to\infty} \left(P_n(a, \varepsilon) - \mathbb{P}\left[\forall i : \max_{t\in(\cup_{j=0}^{n-1} I_j)\cap q_n\mathbb{Z}} U_{t,i}^{(n)} \leq u_{in} \right] \right) = 0. \tag{42}$$

Recalling that $u_{in} = b_n + b_n^{-1}\tau_i$, we find a constant C such that $u_{in}^2 \geq b_n^2 - C$. In the subsequent inequalities, the summation indices t_1 and t_2 take values in $(\cup_{j=0}^{n-1} I_j) \cap q_n\mathbb{Z}$. Set $R(t) = \sup_{h_1,h_2\in B, h_1\neq h_2} r_t(h_1, h_2)$, where $B \subset \mathbb{R}^d$ is a fixed small ball around the origin. By the Berman Inequality, see [14, Theorem 4.2.1], we have, for some constant $K > 0$,

$$\begin{aligned} &\left| P_n(a, \varepsilon) - \mathbb{P}\left[\forall i : \max_{t\in(\cup_{j=0}^{n-1} I_j)\cap q_n\mathbb{Z}} U_{t,i}^{(n)} \leq u_{in} \right] \right| \\ &\quad \leq K \sum_{\substack{t_1 \nsim t_2 \\ i_1,i_2=1,\dots,k}} r_{t_1-t_2}(s_n h_{i_1}, s_n h_{i_2}) \exp\left(-\frac{(u_{i_1 n}^2 + u_{i_2 n}^2)/2}{1 + r_{t_1 t_2}(s_n h_{i_1}, s_n h_{i_2})} \right) \end{aligned}$$

$$\leq K \sum_{\substack{t_1 \not\approx t_2 \\ i_1, i_2 = 1, \ldots, k}} R(t_1 - t_2) \exp\left(-\frac{b_n^2 - C}{1 + R(t_1 - t_2)}\right)$$

$$\leq K e^C k^2 \sum_{t_1 \not\approx t_2} R(t_1 - t_2) \exp\left(-\frac{b_n^2}{1 + R(t_1 - t_2)}\right). \tag{43}$$

Condition (Z3) implies that $R(t) = o(1/\log t)$ as $t \to +\infty$. Further, Condition (Z2) implies that there is $\delta > 0$ such that $R(t_1 - t_2) < 1 - \delta$ provided that $t_1 \not\approx t_2$. These two facts allow us to use Lemma 12.3.1 of [14] to show that the sum on the right-hand side of (43) converges to 0 as $n \to \infty$. This proves (42). To complete the proof of the lemma, recall (41). $\square$

Finally, we are able to complete the proof of the main result of this section.

Proof of Proposition 2.1. Recall from (30) that $\lim_{a \to 0} H_\beta(a) = H_\beta$. Letting $a, \varepsilon \to 0$ in Lemmas 2.4 and 2.5, we obtain that $\lim_{n\to\infty} P_n = e^{-G_{h,\tau}}$. This proves (20), which in combination with (19) yields Proposition 2.1. $\square$

2.2. Tightness

In this section we complete the proof of Theorem 1.1 by showing that the sequence of processes $\{M_n^*\}_{n\in\mathbb{N}}$, where $M_n^*(h) = b_n(M_n(h) - b_n)$, is tight on $C(K)$. Here, $K = [-A, A]^d$ is a fixed d-dimensional cube. We will use some ideas from the proof of Theorem 17 in [13]. However, the main difficulty of our proof, namely handling extremes in continuous time, is not present in [13].

To start with, note that it follows from (10) that the sequence of random variables $\{M_n^*(0)\}_{n\in\mathbb{N}}$ is tight. The continuity modulus of a function $f \in C(K)$ is defined by

$$\omega_\delta(f) = \sup_{\substack{h_1, h_2 \in K \\ |h_1 - h_2| \leq \delta}} |f(h_1) - f(h_2)|, \quad \delta > 0.$$

By a well-known tightness criterion, see [2, Theorem 7.3], the sequence of processes $\{M_n^*\}_{n\in\mathbb{N}}$ is tight on $C(K)$ if the following statement holds: for all $\varepsilon > 0$ and $\varrho > 0$ there exists $\delta > 0$ such that for all sufficiently large n,

$$\mathbb{P}\left[\omega_\delta(M_n^*) > \varrho\right] < 7\varepsilon. \tag{44}$$

The rest of Section 2.2 is devoted to the proof of (44). Let $q_n = b_n^{-2/\beta}$. We set

$$Z_t^*(h) = b_n(Z_t(s_n h) - b_n).$$

(Note that in Section 2.1 we have used a slightly different notation). Further, for $w \in \mathbb{R}$ and $n \in \mathbb{N}$ we define $Z^{w,n} = \{Z_\theta^{w,n}(h); h \in s_n^{-1}D, \theta \in \mathbb{R}\}$ to be the process $\{Z_{q_n\theta}^*(h) - Z_0^*(0); h \in s_n^{-1}D, \theta \in \mathbb{R}\}$ conditioned on the event $Z_0^*(0) = w$. (So, $Z^{w,n}$ is the same as in Section 2.1).

Lemma 2.6. *For $C > 0$, define a random event $E_n(C)$ by*

$$E_n(C) = \left\{\inf_{h\in K} M_n^*(h) \leq -C\right\}.$$

Then there is a sufficiently large $C_1 > 0$ such that for all $n > N$, $\mathbb{P}[E_n(C_1)] < 2\varepsilon$.

Proof. The proof of the analogous statement in [13], see the proof of Theorem 17 there, does not apply in our situation, so we have to use a different method. For $0 < c < C < \infty$ we define auxiliary random events $E_n'(C)$ and $E_n''(c)$ by

$$E_n'(C) = \left\{ \inf_{h \in K} \max_{t \in [0,n] \cap q_n \mathbb{Z}} Z_t^*(h) \le -C \right\},$$

$$E_n''(c) = \left\{ \max_{t \in [0,n] \cap q_n \mathbb{Z}} Z_t^*(0) \in [-c, c] \right\}.$$

It is implicit in [14, Chapter 12] that $\max_{t \in [0,n] \cap q_n \mathbb{Z}} Z_t^*(0)$ has limiting (non-unit) Gumbel distribution (alternatively, see [21, Theorem 2] for an explicit statement). Thus, we may choose c so large that $\mathbb{P}[E_n''(c)] \ge 1 - \varepsilon$ for all n. We have

$$\begin{aligned} \mathbb{P}[E_n(C)] &\le \mathbb{P}[E_n'(C)] \\ &\le \mathbb{P}[E_n'(C) \cap E_n''(c)] + (1 - \mathbb{P}[E_n''(c)]) \\ &\le \mathbb{P}[E_n'(C) \cap E_n''(c)] + \varepsilon. \end{aligned}$$

Further, it is clear that $E_n'(C) \cap E_n''(c) \subset \cup_{t \in [0,n] \cap q_n \mathbb{Z}} A_{t,n}$, where $A_{t,n}$ is a random event defined by

$$A_{t,n} = \left\{ Z_t^*(0) \in [-c, c], \inf_{h \in K} (Z_t^*(h) - Z_t^*(0)) < -(C - c) \right\}.$$

Using this and the stationarity of the process Z in t, we obtain

$$\mathbb{P}[E_n'(C) \cap E_n''(c)] \le \lceil n/q_n \rceil \mathbb{P}[A_{0,n}].$$

To prove the lemma, we have to show that if $C = C_1$ is sufficiently large, then

$$\mathbb{P}[A_{0,n}] < \frac{\varepsilon q_n}{n}. \tag{45}$$

By Part 3 of Lemma 2.1, the family of processes $\{Z_0^{w,n}\}_{w \in [-c,c], n > N}$ is tight on $C(K)$. It follows from the standard tightness criterion, see [2, Theorem 7.3], that for every $\Delta > 0$ we can find a sufficiently large C such that uniformly in $w \in [-c, c]$ and $n > N$,

$$\mathbb{P}\left[\inf_{h \in K} Z_0^{w,n}(h) < -(C - c) \right] < \Delta. \tag{46}$$

Recall that the density of $Z_0^*(0)$ is given by (29). Conditioning on the event $Z_0^*(0) = w$, we obtain

$$\begin{aligned} \mathbb{P}[A_{0,n}] &= \frac{1}{\sqrt{2\pi}} \int_{-c}^{c} \mathbb{P}\left[\inf_{h \in K} (Z_0^*(h) - Z_0^*(0)) < -(C - c) | Z_0^*(0) = w \right] \\ &\quad \times \mathrm{e}^{-(b_n + b_n^{-1} w)^2/2} b_n^{-1} \mathrm{d}w \\ &= \frac{1}{\sqrt{2\pi}} b_n^{-1} \mathrm{e}^{-b_n^2/2} \int_{-c}^{c} \mathbb{P}\left[\inf_{h \in K} Z_0^{w,n}(h) < -(C - c) \right] \mathrm{e}^{-\frac{w^2}{2b_n^2}} \mathrm{e}^{-w} \mathrm{d}w. \end{aligned}$$

It follows from (37) that $b_n^{-1} \mathrm{e}^{-b_n^2/2} = O(q_n/n)$. Using (46), we obtain that for some constant $K = K(c)$,

$$\mathbb{P}[A_{0,n}] \le \frac{K \Delta q_n}{n}.$$

Now choose $C = C_1$ so that $\Delta < \varepsilon/K$. This yields (45) and completes the proof. $\square$

Next, we would like to show that with high probability, only those times t for which $Z_t^*(0) \in [-C, C]$, C large, contribute to the supremum $M_n^*(h) = \sup_{t\in[0,n]} Z_t^*(h)$. For $C > 0$, we define random variables $U_t^{(n)}(h)$ and $M_n^{*,C}(h)$ by

$$U_t^{(n)}(h) = \sup_{\theta\in[0,q_n]} Z_{t+\theta}^*(h), \tag{47}$$

$$M_n^{*,C}(h) = \max\left\{U_t^{(n)}(h) : t \in [0,n] \cap q_n\mathbb{Z},\, Z_t^*(0) \in [-C, C]\right\}. \tag{48}$$

Lemma 2.7. *For $C > 0$, define a random event $F_n(C)$ by*

$$F_n(C) = \left\{\exists h \in K : M_n^*(h) \neq M_n^{*,C}(h)\right\}.$$

Then we can find a sufficiently large $C_2 > 0$ such that for all $n > N$, $\mathbb{P}[F_n(C_2)] \le 4\varepsilon$.

Proof. Let C_1 be given by Lemma 2.6 and take $C > C_1$. For $t \in [0,n] \cap q_n\mathbb{Z}$, define a random event $B_{t,n}$ by

$$B_{t,n} = \left\{Z_t^*(0) \le -C,\, \sup_{h\in K} U_t^{(n)}(h) > -C_1\right\}.$$

Then

$$\mathbb{P}[F_n(C)] \le \mathbb{P}[E_n(C_1)] + \mathbb{P}[M_n^*(0) > C] + \mathbb{P}\left[\cup_{t\in[0,n]\cap q_n\mathbb{Z}} B_{t,n}\right]. \tag{49}$$

By Lemma 2.6, we have $\mathbb{P}[E_n(C_1)] < 2\varepsilon$. It follows from (10) that $\mathbb{P}[M_n^*(0) > C] < \varepsilon$ if C is sufficiently large. So, we concentrate on estimating the third summand on the right-hand side of (49). We use a method from [20, Section D]. It follows from (22) and (24) that for sufficiently large $n > N$ and every $h \in K$, $\theta \in [0,1]$ we have $\mathbb{E}[Z_\theta^{w,n}(h)] < |w|/2$ provided that $|w|$ is large. Conditioning on $Z_0^*(0) = -w$ and recalling that the density of $Z_0^*(0)$ is given by (29), we obtain

$$\begin{aligned}
\mathbb{P}[B_{0,n}] &= \frac{1}{\sqrt{2\pi}}\int_C^\infty \mathbb{P}\left[\sup_{h\in K}\sup_{\theta\in[0,q_n]} Z_\theta^*(h) > -C_1 | Z_0^*(0) = -w\right] \\
&\quad \times \mathrm{e}^{-(b_n - b_n^{-1}w)^2/2} b_n^{-1}\mathrm{d}w \\
&= \frac{1}{\sqrt{2\pi}} b_n^{-1}\mathrm{e}^{-b_n^2/2}\int_C^\infty \mathbb{P}\left[\sup_{h\in K}\sup_{\theta\in[0,1]} Z_\theta^{-w,n}(h) > w - C_1\right]\mathrm{e}^{w}\mathrm{e}^{-\frac{w^2}{2b_n^2}}\mathrm{d}w \\
&\le O\left(\frac{q_n}{n}\right)\int_C^\infty \mathbb{P}\left[\sup_{h\in K}\sup_{\theta\in[0,1]} (Z_\theta^{-w,n}(h) - \mathbb{E}[Z_\theta^{-w,n}(h)]) > \frac{w}{2} - C_1\right]\mathrm{e}^{w}\mathrm{d}w.
\end{aligned} \tag{50}$$

The family of processes $\{Z^{w,n} - \mathbb{E}Z^{w,n}\}_{w\in\mathbb{R},n>N}$ is tight on $C(K \times [0,1])$ by Part 2 of Lemma 2.1. It follows from [2, Theorem 7.3] that we can find a sufficiently large number c_1 such that for all $w \in \mathbb{R}$ and $n > N$,

$$\mathbb{P}\left[\sup_{h\in K}\sup_{\theta\in[0,1]} (Z_\theta^{-w,n}(h) - \mathbb{E}[Z_\theta^{-w,n}(h)]) > c_1\right] < \frac{1}{2}. \tag{51}$$

Recall from (23) that the covariance function of the process $Z^{-w,n}$ does not depend on w. It follows from (26) that there is $\sigma^2 > 0$ such that for all $n > N$,

$$\sup_{h\in K}\sup_{\theta\in[0,1]} \mathrm{Var}[Z_\theta^{-w,n}(h)] \le \sigma^2. \tag{52}$$

Using Borell inequality (see [20, Theorem D.1]) together with (51) and (52), we obtain that for all $w > 2(C_1 + c_1)$ and $n > N$,

$$\mathbb{P}\left[\sup_{h\in K}\sup_{\theta\in[0,1]} (Z_\theta^{-w,n}(h) - \mathbb{E}[Z_\theta^{-w,n}(h)]) > \frac{w}{2} - C_1\right] \le 2\mathrm{e}^{-(\frac{w}{2}-C_1-c_1)^2/(2\sigma^2)}.$$

The right-hand side of the above inequality is not greater than $c_2\mathrm{e}^{-c_3w^2}$ for some constants c_2, c_3. Recalling (50), we obtain that if $C > 2(C_1 + c_1)$, then

$$\mathbb{P}[B_{0,n}] \le O\left(\frac{q_n}{n}\right)\int_C^\infty \mathrm{e}^{-c_3w^2}\mathrm{e}^w \mathrm{d}w.$$

It follows that for a sufficiently large $C = C_2$, $\mathbb{P}[B_{0,n}] < \varepsilon q_n/n$. Recalling (49), we obtain $\mathbb{P}[F_n(C_2)] \le 4\varepsilon$. $\square$

Now we are able complete the proof of (44). Let $D_{t,n}$ be a random event defined by

$$D_{t,n} = \left\{\omega_\delta(U_t^{(n)}) > \varrho, Z_t^*(0) \in [-C_2, C_2]\right\}. \tag{53}$$

Then, using Lemma 2.7, we obtain

$$\mathbb{P}[\omega_\delta(M_n^*) > \varrho] \le \mathbb{P}[F_n(C_2)] + \mathbb{P}[\omega_\delta(M_n^{*,C_2}) > \varrho] \le 4\varepsilon + \lceil n/q_n\rceil \mathbb{P}[D_{0,n}]. \tag{54}$$

We estimate $\mathbb{P}[D_{0,n}]$. To this end, we will use the approximate direct sum structure of the field $Z^{w,n}$ given in Part 1 of Lemma 2.1. More precisely, we would like to use the fact that conditioned on $Z_0^*(0) = w$, we have

$$Z_\theta^*(h) \approx Z_0^*(0) + (Z_\theta^*(0) - Z_0^*(0)) + (Z_0^*(h) - Z_0^*(0)).$$

Thus, we set

$$V_\theta^*(h) = Z_\theta^*(h) - Z_\theta^*(0) - Z_0^*(h) + Z_0^*(0) \tag{55}$$

and

$$V_{\sup}^{(n)} = \sup_{h\in K}\sup_{\theta\in[0,q_n]} V_\theta^*(h),$$
$$V_{\inf}^{(n)} = \inf_{h\in K}\inf_{\theta\in[0,q_n]} V_\theta^*(h).$$

Later we will show that the random variables $V_{\sup}^{(n)}$ and $V_{\inf}^{(n)}$ are in some sense small. By (47), we have

$$\begin{aligned} U_0^{(n)}(h) &= \sup_{\theta\in[0,q_n]} (V_\theta^*(h) + Z_\theta^*(0) + Z_0^*(h) - Z_0^*(0)) \\ &\le Z_0^*(h) - Z_0^*(0) + V_{\sup}^{(n)} + \sup_{\theta\in[0,q_n]} Z_\theta^*(0). \end{aligned}$$

Similarly, we obtain the lower bound

$$U_0^{(n)}(h) \geq Z_0^*(h) - Z_0^*(0) + V_{\inf}^{(n)} + \sup_{\theta\in[0,q_n]} Z_\theta^*(0).$$

Using both bounds, we obtain that for every $h_1, h_2 \in s_n^{-1}K$,

$$|U_0^{(n)}(h_1) - U_0^{(n)}(h_2)| \leq |Z_0^*(h_1) - Z_0^*(h_2)| + V_{\sup}^{(n)} - V_{\inf}^{(n)}.$$

Consequently, we can estimate the continuity modulus of $U_0^{(n)}$ by

$$\omega_\delta(U_0^{(n)}) \leq \omega_\delta(Z_0^*) + V_{\sup}^{(n)} - V_{\inf}^{(n)}.$$

Using the above inequality and recalling (53), we obtain that the random event $D_{0,n}$ is contained in

$$\left(\left\{V_{\sup}^{(n)} > \frac{\varrho}{3}\right\} \cup \left\{V_{\inf}^{(n)} < -\frac{\varrho}{3}\right\} \cup \left\{\omega_\delta(Z_0^*) > \frac{\varrho}{3}\right\}\right) \cap \{Z_0^*(0) \in [-C_2, C_2]\}.$$

Thus, Lemmas 2.8 and 2.9 below imply that $\mathbb{P}[D_{0,n}] \leq 3\varepsilon q_n/n$. Taking into account (54), we obtain (44). This completes the proof of tightness in Theorem 1.1. $\square$

In the rest of the section we state and prove two auxiliary lemmas used above.

Lemma 2.8. *If n is sufficiently large, then*

$$\mathbb{P}\left[V_{\sup}^{(n)} > \frac{\varrho}{3}, Z_0^*(0) \in [-C_2, C_2]\right] \leq \frac{\varepsilon q_n}{n}, \tag{56}$$

$$\mathbb{P}\left[V_{\inf}^{(n)} < -\frac{\varrho}{3}, Z_0^*(0) \in [-C_2, C_2]\right] \leq \frac{\varepsilon q_n}{n}. \tag{57}$$

Proof. For $w \in \mathbb{R}$ and $n \in \mathbb{N}$, let $V^{w,n} = \{V_\theta^{w,n}(h); h \in K, \theta \in [0,1]\}$ be the law of the process $\{V_{q_n\theta}^*(h); h \in K, \theta \in [0,1]\}$ conditioned on $Z_0^*(0) = w$. Recalling (55), we have an equality in distribution

$$V^{w,n} \stackrel{\mathcal{D}}{=} \left\{Z_\theta^{w,n}(h) - Z_\theta^{w,n}(0) - Z_0^{w,n}(h); h \in K, \theta \in [0,1]\right\}.$$

The asymptotic direct sum structure of the process $Z^{w,n}$ given in Part 1 of Lemma 2.1 implies that $V^{w,n}$ converges to 0 weakly on $C(K \times [0,1])$, the convergence being uniform as long as w stays bounded. It follows that for every $\Delta > 0$ there is $N(\Delta)$ such that for all $n > N(\Delta)$ and $w \in [-C_2, C_2]$,

$$\mathbb{P}\left[\sup_{h\in K}\sup_{\theta\in[0,1]} V_\theta^{w,n}(h) > \frac{\varrho}{3}\right] < \Delta.$$

The probability on the left-hand side of (56) can be written as

$$\begin{aligned}\text{LHS of (56)} &= \frac{1}{\sqrt{2\pi}}\int_{-C_2}^{C_2} \mathbb{P}\left[\sup_{h\in K}\sup_{\theta\in[0,1]} V_\theta^{w,n}(h) > \frac{\varrho}{3}\right] \mathrm{e}^{-(b_n+b_n^{-1}w)^2/2} b_n^{-1}\mathrm{d}w \\ &\leq \Delta \cdot O\left(\frac{q_n}{n}\right) \cdot \int_{-C_2}^{C_2} \mathrm{e}^{-w}\mathrm{e}^{-\frac{w^2}{2b_n^2}}\mathrm{d}w \\ &\leq \Delta \cdot O\left(\frac{q_n}{n}\right).\end{aligned}$$

Thus, if we choose Δ is small enough, then (56) holds for all $n > N(\Delta)$. The proof of (57) is analogous. $\square$

Lemma 2.9. *If $\delta > 0$ is sufficiently small, then*

$$\mathbb{P}\left[\omega_\delta(Z_0^*) > \frac{\varrho}{3}, Z_0^*(0) \in [-C_2, C_2]\right] < \frac{\varepsilon q_n}{n}. \tag{58}$$

Proof. By Part 3 of Lemma 2.1, the family of processes $\{Z_0^{w,n}\}_{w\in[-C_2,C_2],n>N}$ is tight on $C(K)$. It follows from [2, Theorem 7.3] that for every $\Delta > 0$ we can choose $\delta > 0$ so small that

$$\mathbb{P}\left[\omega_\delta(Z_0^{w,n}) > \frac{\varrho}{3}\right] < \Delta.$$

Conditioning on $Z_0^*(0) = w$, we obtain

$$\begin{aligned}\text{LHS of (58)} &= \frac{1}{\sqrt{2\pi}} \int_{-C_2}^{C_2} \mathbb{P}\left[\omega_\delta(Z_0^{w,n}) > \frac{\varrho}{3}\right] \mathrm{e}^{-(b_n+b_n^{-1}w)^2/2} b_n^{-1} \mathrm{d}w \\ &\le \Delta \cdot O\left(\frac{q_n}{n}\right) \cdot \int_{-C_2}^{C_2} \mathrm{e}^{-w} \mathrm{e}^{-\frac{w^2}{2b_n^2}} \mathrm{d}w \\ &\le \Delta \cdot O\left(\frac{q_n}{n}\right).\end{aligned}$$

Choosing Δ and then δ small enough, we obtain (58). $\square$

3. Proof of Theorem 1.2

In this section we prove Theorem 1.2. The idea is to approximate the empirical process α_t by a normalized Kiefer process $t^{-1/2}K_t$, and then to apply Theorem 1.1 to an appropriate transformation of K_t. The Kiefer process is a zero-mean Gaussian process $K = \{K_t(h); h \in [0, 1], t \ge 0\}$ whose covariance function is given by

$$\mathbb{E}[K_{t_1}(h_1), K_{t_2}(h_2)] = \min(t_1, t_2)(\min(h_1, h_2) - h_1 h_2). \tag{59}$$

In other words, K is a Brownian bridge in the space direction, and a Brownian motion in the time direction. Let $\sigma^2(h) = h(1 - h)$ be the variance of $K_1(h)$.

We start by proving a lemma which is an analogue of Theorem 1.2 with the empirical process replaced by the normalized Kiefer process. Fix $h_0 \in (0, 1)$ and define

$$M_n(h) = \frac{1}{\sigma(h_0)} \sup_{t\in[1,n]} t^{-1/2} K_t(h_0 + \tilde{s}_n h). \tag{60}$$

Lemma 3.1. *The process $\tilde{b}_n(M_n - \tilde{b}_n)$ converges as $n \to \infty$ to the drifted Brown–Resnick process η defined as in Theorem 1.2 in the sense of finite-dimensional distributions.*

Proof. The normalized Kiefer process $t^{-1/2}K_t$ is not time-stationary. To prove the lemma, we will apply Theorem 1.1 to a time-stationary transformation of the Kiefer process. We define a process $Z = \{Z_t(h); h \in (0, 1), t \in \mathbb{R}\}$ by

$$Z_t(h) = \sigma^{-1}(h)\mathrm{e}^{-t/2} K_{\mathrm{e}^t}(h). \tag{61}$$

Then the covariance function of Z is given by

$$\mathbb{E}[Z_{t_1}(h_1), Z_{t_2}(h_2)] = \mathrm{e}^{-|t_1-t_2|/2} \cdot \left(\frac{\min(h_1, h_2) - h_1 h_2}{\sigma(h_1)\sigma(h_2)} \right).$$

Hence, in the space direction, Z is a normalized Brownian bridge, whereas the time evolution of Z is described by an Ornstein–Uhlenbeck process.

An easy calculation shows that the translated process $\{Z_t(h_0+h); h \in (-h_0, 1-h_0), t \in \mathbb{R}\}$ satisfies Conditions (Z1)–(Z3) with

$$\alpha = \beta = 1, \qquad c_\alpha = 1/(2\sigma^2(h_0)), \qquad c_\beta = 1/2.$$

Define s_n and b_n as in (7) and (8), and recall that $\tilde{s}_n$ and $\tilde{b}_n$ were defined in (12) and (13). Since the Pickands constant H_1 equals 1, see [14, Section 12.2], we have $\tilde{b}_n = b_{\log n}$ and $\tilde{s}_n = s_{\log n}$. Let

$$M_n'(h) = \sup_{t\in[0,n]} Z_t(h_0 + s_n h). \tag{62}$$

Applying Theorem 1.1, we obtain that the process $b_n(M_n' - b_n)$ converges as $n \to \infty$ to the Brown–Resnick process η_1.

To prove the lemma, we need to compare M_n to M_n'. It follows from (60)–(62) that

$$M_n(h) = \frac{\sigma(h_0 + \tilde{s}_n h)}{\sigma(h_0)} M'_{\log n}(h).$$

Hence, we may write

$$\tilde{b}_n(M_n(h) - \tilde{b}_n) = \frac{\sigma(h_0 + \tilde{s}_n h)}{\sigma(h_0)} \tilde{b}_n(M'_{\log n}(h) - \tilde{b}_n) + \tilde{b}_n^2 \left(\frac{\sigma(h_0 + \tilde{s}_n h)}{\sigma(h_0)} - 1 \right). \tag{63}$$

An easy calculation shows that

$$\tilde{b}_n^2 \left(\frac{\sigma(h_0 + \tilde{s}_n h)}{\sigma(h_0)} - 1 \right) = (1 - 2h_0)h + o(1), \quad n \to \infty. \tag{64}$$

Recall that $\tilde{b}_n = b_{\log n}$. It follows from (63) and (64) that, as $n \to \infty$,

$$\tilde{b}_n(M_n(h) - \tilde{b}_n) = (1 + o(1)) b_{\log n}(M'_{\log n}(h) - b_{\log n}) + (1 - 2h_0)h + o(1).$$

To complete the proof, recall that the process $b_{\log n}(M'_{\log n} - b_{\log n})$ converges as $n \to \infty$ to the Brown–Resnick process η_1 by Theorem 1.1. □

Since we will be able to approximate α_t by $t^{-1/2}K_t$ for sufficiently large t only, we need to show that the initial segment of the empirical (Kiefer) process asymptotically does not contribute to L_n (M_n, respectively). This is done in the next two lemmas. Recall that L_n and M_n were defined in (14) and (60). Given $1 \le k \le l$, we define

$$L_n^{(k,l)}(h) = \frac{1}{\sigma(h_0)} \sup_{t\in[k,l]} \alpha_t(h_0 + \tilde{s}_n h), \tag{65}$$

$$M_n^{(k,l)}(h) = \frac{1}{\sigma(h_0)} \sup_{t\in[k,l]} t^{-1/2} K_t(h_0 + \tilde{s}_n h). \tag{66}$$

Lemma 3.2. *Let $k_n = \log n$. Then for every $h \in \mathbb{R}$,*

$$\lim_{n\to\infty} \tilde{b}_n(L_n^{(1,k_n)}(h) - \tilde{b}_n) = \lim_{n\to\infty} \tilde{b}_n(M_n^{(1,k_n)}(h) - \tilde{b}_n) = -\infty \quad \textit{a.s.}$$

Proof. By [5, Theorem 5.1.1] and [5, Corollary 1.15.1],

$$\limsup_{t\to\infty} \frac{1}{\sqrt{\log_2 t}} \sup_{h\in(0,1)} \alpha_t(h) = \limsup_{t\to\infty} \frac{1}{\sqrt{\log_2 t}} \sup_{h\in(0,1)} t^{-1/2} K_t(h) = \frac{1}{\sqrt{2}}.$$

Let c be arbitrary such that $c > 1/(\sigma(h_0)\sqrt{2})$. Let $h \in \mathbb{R}$. It follows that for all sufficiently large n,

$$L_n^{(1,k_n)}(h) \le c\sqrt{\log_2 k_n}, \qquad M_n^{(1,k_n)}(h) \le c\sqrt{\log_2 k_n}.$$

The statement of the lemma follows by noting that $c\sqrt{\log_2 k_n} < \tilde{b}_n - 1$ for n large enough. □

Lemma 3.3. *Let $k_n = \log n$. Then the process $\tilde{b}_n(L_n - \tilde{b}_n)$ converges as $n \to \infty$ to η in the sense of finite-dimensional distributions iff the process $\tilde{b}_n(L_n^{(k_n,n)} - \tilde{b}_n)$ does. A similar statement holds for the processes $\tilde{b}_n(M_n - \tilde{b}_n)$ and $\tilde{b}_n(M_n^{(k_n,n)} - \tilde{b}_n)$.*

Proof. Let $d \in \mathbb{N}$, and fix some $h_1, \dots, h_d \in \mathbb{R}$ and $\tau_1, \dots, \tau_d \in \mathbb{R}$. For $1 \le k \le l$ let

$$P_n^{(k,l)} = \mathbb{P}[\tilde{b}_n(L_n^{(k,l)}(h_i) - \tilde{b}_n) \le \tau_i \text{ for all } i = 1, \dots, d].$$

Note that $L_n = \max(L_n^{(1,k_n)}, L_n^{(k_n,n)})$. It follows that

$$P_n^{(1,n)} \le P_n^{(k_n,n)} \le P_n^{(1,n)} + \sum_{i=1}^{d} \mathbb{P}\left[\tilde{b}_n(L_n^{(1,k_n)}(h_i) - \tilde{b}_n) > \tau_i\right]. \tag{67}$$

By Lemma 3.2, the sum on the right-hand side converges to 0 as $n \to \infty$. Thus, $P_n^{(1,n)}$ converges as $n \to \infty$ to some limit iff $P_n^{(k_n,n)}$ converges to the same limit. The statement of the lemma follows. □

Proof of Theorem 1.2. By a strong approximation theorem of Komlós–Major–Tusnády [5, Theorem 4.4.3], we can construct on some probability space an empirical process and a Kiefer process such that

$$\sup_{h\in[0,1]} |\alpha_t(h) - t^{-1/2} K_t(h)| = O\left(\frac{\log^2 t}{\sqrt{t}}\right) \quad \text{a.s. as } t \to \infty.$$

Note that with $k_n = \log n$, we have $\log^2 k_n/\sqrt{k_n} = o(1/\tilde{b}_n)$ as $n \to \infty$. It follows that

$$\lim_{n\to\infty} \tilde{b}_n \sup_{h\in[0,1]} \left| \sup_{t\in[k_n,n]} \alpha_t(h) - \sup_{t\in[k_n,n]} t^{-1/2} K_t(h) \right| = 0 \quad \text{a.s.} \tag{68}$$

Recall that the processes $L_n^{(k_n,n)}$ and $M_n^{(k_n,n)}$ were defined in (65) and (66). Then it follows from (68) that for every $h \in \mathbb{R}$,

$$\lim_{n\to\infty} \left(\tilde{b}_n(L_n^{(k_n,n)}(h) - \tilde{b}_n) - \tilde{b}_n(M_n^{(k_n,n)}(h) - \tilde{b}_n)\right) = 0 \quad \text{a.s.} \tag{69}$$

It follows from Lemmas 3.1 and 3.3 that the process $\tilde{b}_n(M_n^{(k_n,n)} - \tilde{b}_n)$ converges as $n \to \infty$ to the drifted Brown–Resnick process η in the sense of finite-dimensional distributions. By (69), this implies that the process $\tilde{b}_n(L_n^{(k_n,n)} - \tilde{b}_n)$ converges to η as well. Again using Lemma 3.3, we see that the process $\tilde{b}_n(L_n - \tilde{b}_n)$ converges as $n \to \infty$ to η. This completes the proof. □

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